APPROXIMATE CALCULATION OF THE UNSTEADY MELTING OF VITREOUS MATERIALS

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This paper deals with the problem of unsteady melting of a vitreous layer due to aerodynamic heating in the absence of self-similar solutions. Aerodynamic heating and friction are regarded as known functions of the surface temperature, the longitudinal coordinate, and the time.

Statement of the problem. The flow of a viscous film of melt is described by the well-known boundary-layer equations

$$\frac{\partial (r^k u)}{\partial x} + \frac{\partial (r^k v)}{\partial y} = 0 \tag{1}$$

($k = 1$ for axisymmetric flow and $k = 0$ for plane flow),

$$\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) + \rho X - \frac{\partial p}{\partial x} = 0, \tag{2}$$

$$\rho c\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\lambda \frac{\partial T}{\partial y}\right). \tag{3}$$

The x axis is directed along the surface of the solid at the initial instant and the y axis is directed along the normal to this surface into the solid. Henceforth we will regard ρ, c, and λ as constants, X and p as known functions of x, and μ as an unknown function of the temperature. For vitreous materials usually $\mu = \exp[(a/T)^m + b]$.

The simplifications made in Eqs. (2) and (3) are generally accepted in the solution of similar problems. They are justified in view of the low velocity of flow of the film of melt.

Boundary conditions are imposed on the moving surface $y = y_W(x, t)$,

$$\lambda \frac{\partial T}{\partial y} = -q_w, \quad \mu \frac{\partial u}{\partial y} = -\tau_w, \tag{4}$$

where q_W and τ_W are assumed to be known functions of the surface temperature, the longitudinal coordinate, and the time.

In the absence of evaporation of the melt, $y_W(x, t)$ is given by the equation

$$\frac{\partial y_w}{\partial t} + u_w \frac{\partial y_w}{\partial x} = v_w. \tag{5}$$

In the case of a considerable thickness of layer we can solve the problem as for a semi-infinite body and impose the boundary condition

$$\text{when} \quad T \leqslant T_m \quad u = v = 0, \tag{6}$$

where T_m is an "arbitrary melting point," which is introduced for vitreous materials so that the mathematical statement of the problem is approximately correct. The reasons for the introduction of an "arbitrary melting point" are given in [2].

The initial conditions are

$$\text{when} \quad t = 0 \quad T = T_0(x, y), \quad y_w = 0. \tag{7}$$

In addition, we must impose conditions when $x = 0$. For instance, at the frontal point of a blunt body

$$\text{when} \quad x = 0 \quad \frac{\partial T}{\partial x} = 0, \quad \frac{\partial y_w}{\partial x} = 0. \tag{8}$$

Method of solution. We assume that function $T_0(x, y)$ is such that when $y \to \infty$, $T_0(x, y) \to T_\infty$ (T_∞ = const) and $\partial T_0/\partial y \to 0$.

We can then show that the required function $T(x, y, t)$ for any fixed t has the same properties when $y \to \infty$.

Introducing the new required function $\Theta = T - T_\infty$ and then integrating Eq. (3) with respect to y from y_W to infinity, we obtain the integral relationship

$$\frac{\partial A}{\partial t} + \frac{1}{r^k}\frac{\partial (r^k B)}{\partial x} = \frac{q_w}{\rho c}, \tag{9}$$

where

$$A = \int_{y_w}^{\infty} \Theta\, dy, \quad B = \int_{y_w}^{\infty} u\Theta\, dy.$$

We approximate the temperature distribution by an exponential dependence on y,

$$\Theta = \Theta_w(x, t)\exp[-(y - y_w)/\delta(x, t)]. \tag{10}$$

It is obvious that the above conditions at infinity are fulfilled in this case. Using the first of the conditions (4) we obtain a relationship between Θ_W and δ

$$\delta = \lambda\Theta_w/q_w,$$

and to determine $\Theta_W(x, t)$ we use Eq. (9).

To determine the position and velocity of displacement of the surface of the liquid film we use Eq. (5), which can be converted, by reference to (1), to the form

$$\frac{\partial y_w}{\partial t} = \frac{1}{r^k}\frac{\partial (r^k C)}{\partial x}, \tag{11}$$

where

$$C = \int_{y_w}^{\infty} u\, dy.$$

To calculate B and C we need to know the velocity u. From Eq. (2) we can obtain

$$u=\int\limits_{y_m}^{y}\frac{1}{\mu}\left[\left(\frac{\partial p}{\partial x}-\rho X\right)(y-y_w)-\tau_w\right]dy$$

($y = y_m(x, t)$ is the surface on which $T = T_m$).

If the relationship between viscosity and temperature is approximated by a power relationship

$$\mu/\mu_w=(\Theta/\Theta_w)^{-n}, \qquad (12)$$

then, in view of (10), the viscosity distribution in the layer of melt will take the form

$$\mu=\mu_w\exp\left(n\,\frac{y-y_w}{\delta}\right),$$

and, hence,

$$u=\frac{\delta}{\mu_w n}\left[\tau_w-\left(\frac{\partial p}{\partial x}-\rho X\right)\left(\frac{\delta}{n}-y+y_w\right)\right]\times$$
$$\times\exp\left(-n\,\frac{y-y_w}{\delta}\right)-\frac{\delta}{\mu_m n}\times$$
$$\times\left[\tau_w-\left(\frac{\partial p}{\partial x}-\rho X\right)\left(\frac{\delta}{n}-y_m+y_w\right)\right]. \qquad (13)$$

Since the viscosity $\mu_m = \mu(T_m)$ is high (this is one of the conditions of selection of T_m), the second term in formula (13) can be neglected, which is equivalent to replacement of condition (6) by the condition

$$\text{when}\quad y\to\infty\quad u\to 0,\quad v\to 0.$$

Such a replacement can be carried out only after an approximation of the actual law of variation of the viscosity by the approximate relationship (12). In the initial statement of the problem the condition when $y \to \infty$ does not ensure a finite value of the velocity for an arbitrarily assigned T_∞.

Having the distribution of velocity (13) and temperature (10), we easily obtain

$$A=\Theta_w\delta=\lambda\,\frac{\Theta_w^2}{q_w},$$
$$B=\frac{\Theta_w\delta^2}{\mu_w n(n+1)}\left[\tau_w-\delta\,\frac{2n+1}{n(n+1)}\left(\frac{\partial p}{\partial x}-\rho X\right)\right],$$
$$C=\frac{\delta^2}{\mu_w n^2}\left[\tau_w-\frac{2\delta}{n}\left(\frac{\partial p}{\partial x}-\rho X\right)\right].$$

In the deduction of these formulas terms of the order $1/\mu_m$ were neglected.

It should be noted that our solution depends on n, the index of the power in formula (12), and hence this gives rise to the question of the best approximation of the viscosity. It is obvious that n can depend on the temperature range in which the approximation is made, and hence may depend on x and t. Using the correspondence of the true value of the viscosity and the approximate one at points Θ_w and Θ_1, we obtain

$$n=\ln\frac{\mu(\Theta_1)}{\mu(\Theta_w)}\Big/\ln\frac{\Theta_w}{\Theta_1}. \qquad (14)$$

The choice of Θ_1 is arbitrary within wide limits. The best choice is $\Theta_1 = \Theta_w/2$, where the viscosity is fairly well approximated in a wide temperature range. The limiting case $\Theta_1 \to \Theta_w$, used in [3], gives a good approximation only close to the surface of the liquid film, which leads to a large error in the calculation.

Thus, the problem has been reduced to the solution of Eqs. (9) and (11) with the initial conditions

$$\text{when}\ t=0\quad \Theta_w=T_0(x,\,0)-T_\infty,\ y_w=0,$$
$$\text{and when}\ x=0\quad \frac{\partial\Theta_w}{\partial x}=0,\quad \frac{\partial y_w}{\partial x}=0.$$

Instead of the condition for Θ_w we can write the condition for A,

$$\text{when}\ t=0\quad A=A_0,$$

where

$$A_0=\int\limits_0^\infty[T_0(x,\,y)-T_\infty]\,dy.$$

Solution of heat-conduction equation and calculation of melting at the frontal point of a blunt body. We will consider the application of the above method in particular cases for comparison with known solutions.

Heat-conduction equation. In the solution of the heat-conduction equation by the integral method we obtain the equation for the determination of $\Theta_w(t)$

$$\frac{d}{dt}\left(\lambda\,\frac{\Theta_w^2}{q_w}\right)=\frac{q_w}{\rho c}.$$

If q_w = const and $\Theta_w(0) = 0$, then

$$\Theta_w=\frac{q_w}{\sqrt{\rho c\lambda}}\sqrt{t}.$$

If $q_w = \alpha(\Theta_{00} - \Theta_w)$ and $\Theta_w(0) = 0$, then

$$\frac{\alpha^2}{\rho c\lambda}\,t=\ln\left(1-\frac{\Theta_w}{\Theta_{00}}\right)+\frac{1}{2}\left[\frac{1}{(1-\Theta_w/\Theta_{00})^2}-1\right].$$

The corresponding exact solutions [4] have the form

$$\Theta_w=\frac{2}{\sqrt{\pi}}\,\frac{q_w}{\sqrt{\rho c\lambda}}\sqrt{t}$$

for q_w = const and

$$1-\frac{\Theta_w}{\Theta_{00}}=\left[1-\frac{2}{\sqrt{\pi}}\int\limits_0^{\frac{\alpha}{\sqrt{\rho c\lambda}}\sqrt{t}}\exp(-z^2)\,dz\right]\exp\left(\frac{\alpha^2 t}{\rho c\lambda}\right)$$

for $q_w = \alpha(\Theta_{00} - \Theta_w)$.

The figure shows the results of calculation from these formulas with the following data: ρ = 2100 kg/m^3, c = 1210 J/kg·deg, λ = 10.6 W/m·deg. Curves 6 and 7 are the exact and approximate solutions for $q_w = 4.35\cdot10^6$ W/m^2, and curves 1 and 2 are for $q_w = 1740\,(6700 - \Theta_w)$ W/m^2. In both cases the error of the approximate solution does not exceed 12%.

Melting at frontal point. At the frontal point of a blunt body $\partial p/\partial x = p''x$, $\tau_w = \tau'_w x$, $r = x$, $\partial\Theta/\partial x = 0$, and in the absence of mass forces (X = 0) we have

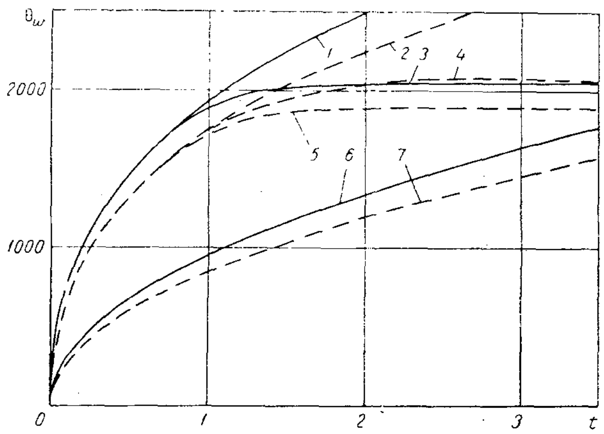
$$\frac{1}{r^k}\,\frac{\partial(r^kB)}{\partial x}=\frac{(k+1)\Theta_w\delta^2}{\mu_w n(n+1)}\left[\tau'_w-p''\delta\,\frac{2n+1}{n(n+1)}\right].$$

In the case $q_W = \alpha(\Theta_{00} - \Theta_W)$, Eqs. (9) and (11) take the form

$$\frac{\lambda^2}{\alpha^2}\frac{(2\Theta_{00}-\Theta_w)\Theta_w}{(\Theta_{00}-\Theta_w)^3}\frac{d\Theta_w}{dt} = \frac{\lambda}{\rho c} - \frac{(k+1)\delta^3}{\mu_w n(n+1)}\left[\tau_w' - p''\delta\frac{2n+1}{n(n+1)}\right], \quad (15)$$

$$\frac{dy_w}{dt} = \frac{(k+1)\delta^2}{\mu_w n^2}\left[\tau_w' - 2p''\frac{\delta}{n}\right]. \quad (16)$$

The calculation was carried out for $\tau_W' = 7190\ N/m^3$, $p'' = -2.885\cdot10^7\ N/m^4$, $T_\infty = 300°\ K$, $\mu = \exp[(4840/T)^{1.612} - 2.3]\ N\cdot sec/m^2$ (the rest of the data were the same as in the solution of the heat-conduction equation).



Surface temperature Θ_W, as function of time t, sec: 1) from heat-conduction equation $q_W = \alpha(\Theta_{00} - \Theta_W)$, exact solution; 2) approximate solution; 3) melting, exact solution; 4) approximate solution, n from (14); 5) approximate solution, n from (17); 6) from heat-conduction equation $q_W = \text{const}$, exact solution; 7) approximate solution.

In one form of the calculation we used formula (14), where $\Theta_1 = \Theta_W/2$, and in the second form we used the formula

$$n = 1.612(4840/T)^{1.612}\,\Theta_w/(\Theta_w + T_\infty), \quad (17)$$

obtained from (14) by the limiting transition when $\Theta_1 \to \Theta_W$. Curves 4 and 5 in Fig. 1 correspond to these forms. A comparison of the results of calculation with the exact solution (curve 3) shows that in the choice of n we must obtain a good approximation of the viscosity over the whole range of temperature variation, and not only close to the surface.

Steady-state melting. To obtain the surface temperature in steady-state melting at the frontal point we need merely equate the right side of (15) to zero,

$$\frac{\lambda}{\rho c} - \frac{(k+1)\delta^3}{\mu_w n(n+1)}\left[\tau_w' - p''\delta\frac{2n+1}{n(n+1)}\right] = 0.$$

It is interesting to compare this equation with the surface temperature in [1], which in our symbols has the form

$$\frac{\lambda}{\rho c} - \frac{(k+1)\delta^3}{\mu_w n^2}\left[\tau_w' - p''\delta\frac{2}{n}\right] = 0.$$

The rate of melting is calculated, as in this work, from formula (16), but in view of the difference in the temperature determination the rate of melting obtained is different from that calculated by our method. Calculation with the above data by the method of [1] gave $\Theta_W = 2030°$ K, $dy_W/dt = 1.57\cdot10^{-3}$ m/sec, and from the method of this work $\Theta_W = 2060°$ K, $dy_W/dt = 1.83\cdot10^{-3}$ m/sec, whereas the exact solution is $\Theta_W = 2047°$ K, $dy_W/dt = 1.75\cdot10^{-3}$ m/sec.

The conducted calculations show that the above-expounded integral method can give results which are in satisfactory agreement with the exact solution. This method also enables a fairly simple calculation of unsteady melting of a vitreous layer not only in the vicinity of the frontal point, but also on the side surface.

NOTATION

t is the time; u, v are the projections of velocity on x and y axes; X is the projection of mass forces on x axis; T is the temperature; μ is the viscosity; λ is the thermal conductivity; c is the specific heat; ρ is the density; p is the pressure; q is the heat flux; τ is the friction stress; r is the distance from outline of body to axis of symmetry. The subscript ω refers to values on the melt surface.

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